

LIOUVILLIAN AND ANALYTIC FIRST INTEGRALS FOR THE BRUSSELATOR SYSTEM

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ABSTRACT. We characterize the Liouvillian and analytic first integrals for the polynomial differential systems of the form $x' = a - (b + 1)x + x^2y$, $y' = bx - x^2y$, with $a, b \in \mathbb{R}$, called the Brusselator differential systems.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

One of the more classical problems in the qualitative theory of planar differential systems depending on parameters is to characterize the existence or not of first integrals.

We consider the system

$$(1) \quad x' = a - (b + 1)x + x^2y, \quad y' = bx - x^2y$$

that we call the *Brusselator*, where x and y are complex variables and the prime denotes derivative with respect to the time t , which can be either real or complex. Such differential systems appear in several branches of the sciences, mainly in chemistry since it studies a certain chemical reaction (see [7, 9, 16] for details).

Let $U \subset \mathbb{C}^2$ be an open and dense set in \mathbb{C}^2 . We say that the non-constant function $H: \mathbb{C}^2 \rightarrow \mathbb{C}$ is a first integral of the polynomial vector field χ on U , if $H(x(t), y(t)) = \text{constant}$ for all values of t for which the solution $(x(t), y(t))$ of χ is defined on U . Clearly H is a first integral of χ on U if and only if $\chi H = 0$ on U .

An *analytic first integral* is a first integral which is an analytic function. As a particular case, a first integral which is a polynomial is called a *polynomial first integral*.

A *Liouvillian first integral* is a first integral H which is a Liouvillian function, that is, roughly speaking which can be obtained “by quadratures” of elementary functions. For a precise definition see [20]. The study of the Liouvillian first integrals is a classical problem of the integrability theory of the differential equations which goes back to Liouville, see for details again [20].

As far as we know the Liouvillian and analytic first integrals of some multi-parameter family of planar polynomial differential systems has only been classified for few differential systems, see for instance [3, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18].

We first study system (1) with $a = 0$. In this case changing the independent variable t as follows $x dt = d\tau$, it becomes

$$(2) \quad \dot{x} = -b - 1 + xy, \quad \dot{y} = b - xy,$$

where the dot denotes derivative with respect to τ .

Let F be the Kummer confluent hypergeometric function, and H_b be the Hermite function, when b is a positive integer provides the Hermite polynomial of degree n . For more details on these two functions see [1].

2010 *Mathematics Subject Classification.* 34C35, 34D30.

Key words and phrases. Darboux polynomial, exponential factor, Liouvillian first integral, analytic first integral, Brusselator differential systems.

Proposition 1. *System (2) has the analytic first integral*

$$(3) \quad \frac{(b+1)(y+x) F\left(-\frac{b}{2}; \frac{3}{2}; \frac{(y+x)^2}{2}\right) - x F\left(-\frac{b}{2}; \frac{1}{2}; \frac{(y+x)^2}{2}\right)}{\sqrt{2}bH_{b-1}\left(\frac{y+x}{\sqrt{2}}\right) - yH_b\left(\frac{y+x}{\sqrt{2}}\right)}$$

It is easy to check that the function given in (3) is an analytic first integral of system (1) with $a = 0$ defined in an open and dense set of \mathbb{C}^2 .

Theorem 2. *The Brusselator system (1) with $a \neq 0$ has no analytic first integrals.*

Theorem 2 is proved in Section 2.

It follows from Theorem 2 that system (1) with $a \neq 0$ has no Liouvillian first integrals (otherwise it would have analytic first integrals since the last class include in particular the Liouvillian first integrals). However when $a = 0$, it could be that for some values of b , the first integral given in (3) could be of Liouvillian type and that system (2) could have a Liouvillian first integral. We will see that this can not happen. So, from now on we restrict to system (2).

Let $h(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C}$. As usual $\mathbb{C}[x, y]$ denotes the ring of all complex polynomials in the variables x and y . We say that $h = 0$ is an *invariant algebraic curve* of the vector field X associated to system (2) if it satisfies

$$(4) \quad (-b-1+xy)\frac{\partial h}{\partial x} + (b-xy)\frac{\partial h}{\partial y} = Kh,$$

the polynomial $K = K(x, y) \in \mathbb{C}[x, y]$ is called the *cofactor* of $h = 0$ and has degree at most 1. We also say that h is a *Darboux polynomial* of system (1). Note that a polynomial first integral is a Darboux polynomial with zero cofactor.

The invariant algebraic curves are important because a sufficient number of them forces the existence of a first integral. This result is the basis of the Darboux theory of integrability, see for instance [3, 5, 8]. In the next result we characterize all the algebraic invariant curves for system (2).

Proposition 3. *The following statements hold for system (2).*

- (a) *It has no Darboux polynomials with zero cofactor.*
- (b) *It has an irreducible Darboux polynomial $h = h(x, y)$ with non-zero cofactor if and only if $b = 0$, and in this case $h = y$ and the cofactor is $-x$.*

Proposition 3 is proved in Section 3.

An *exponential factor* E of system (1) is a function of the form $E = \exp(g/h) \notin \mathbb{C}$ with $g, h \in \mathbb{C}[x, y]$ coprime satisfying that

$$(5) \quad (-b-1+xy)\frac{\partial E}{\partial x} + (b-xy)\frac{\partial E}{\partial y} = LE,$$

for some polynomial $L = L(x, y)$ of degree at most 1, called the *cofactor* of E .

The existence of exponential factors $\exp(g/h)$ is due to the fact that the multiplicity of the invariant algebraic curve $h = 0$ is larger than 1, and when h is constant, the existence of the exponential factor is due to the multiplicity of the straight line at infinity in the projective plane, for more details see [5].

Proposition 4. *System (2) has two exponential factors modulo constants: e^{x+y} with cofactor -1 and $e^{(x+y)^2}$ with cofactor $-2(x+y)$.*

Proposition 4 is proved in Section 4.

Theorem 5. *System (2) has a Liouvillian first integral if and only if $b = 0$ and in this case the Liouvillian first integral is*

$$\frac{1}{y} \left(\sqrt{2\pi} y \operatorname{Erfi} \left(\frac{-x-y}{\sqrt{2}} \right) + 2e^{\frac{1}{2}(x+y)^2} \right),$$

where $\operatorname{Erfi}(z)$ is the imaginary error function, for more details see [1].

Theorem 5 is proved in Section 5.

As a corollary of Theorems 2 and 5 we have the following result.

Corollary 6. *System (1) has a analytic first integral if and only if $a = 0$. Moreover such first integral is Liouvillian if and only if $b = 0$.*

2. PROOF OF THEOREM 2

We will show that system (1) has no analytic first integrals. To do so, we need the following auxiliary result, it is due to Poincaré in [19], see also [6] for a direct proof. Through the paper \mathbb{Z}^+ will denote the set of non-negative integers.

Theorem 7. *Assume that the eigenvalues of the linear part at some singular point p of the vector field χ , λ_1 and λ_2 are non-zero and that they do not satisfy any resonance condition of the form*

$$\lambda_1 k_1 + \lambda_2 k_2 = 0 \quad \text{for } k_1, k_2 \in \mathbb{Z}^+ \quad \text{with } k_1 + k_2 > 0.$$

Then the vector field χ has no analytic first integrals defined in a neighborhood of p .

Proof of Theorem 2. We compute the singular points of system (1) and we get that these are $p = (a, (b + a^2)/a)$. Computing the eigenvalues λ_1 and λ_2 of the Jacobian matrix of system (1) at the fixed point p we get that they are

$$\lambda_{1,2} = \frac{1}{2}(b - 1 - a^2 \pm \sqrt{-4a^2 + (1 + a^2 - b)^2}).$$

Suppose that there exists $k_1, k_2 \in \mathbb{Z}^+$ such that $k_1 \lambda_1 + k_2 \lambda_2 = 0$. Note that by Theorem 7 if such integers do not exist we are done. Then $\lambda_1 = -\alpha \lambda_2$ with α a positive rational, and hence in particular $\lambda_1 \lambda_2 = -\alpha \lambda_2^2 < 0$. But we have that $\lambda_1 \lambda_2 = a^2 > 0$. This completes the proof of Theorem 2. \square

3. PROOF OF PROPOSITION 3

We first prove the part of the proposition concerning the existence of polynomial first integrals.

Lemma 8. *System (2) has no polynomial first integrals.*

Proof. We introduce the change of variables

$$(6) \quad X = x, \quad Y = y + X$$

and system (2) becomes

$$(7) \quad X' = -b - 1 + XY - X^2, \quad Y' = -1.$$

Since the change is linear, clearly it is equivalent to look for polynomial first integrals $h(x, y)$ of system (2) that to look for polynomial first integrals $\bar{h}(X, Y) = h(x, y)$ of system (7). We write \bar{h} as a polynomial in the variable X , i.e.

$$(8) \quad \bar{h} = \sum_{j=0}^n \bar{h}_j(Y) X^j,$$

where each \bar{h}_j are polynomials in the variable Y . Since \bar{h} is a polynomial first integral we can assume that $\bar{h}_n \neq 0$ and $n > 0$, because if $n = 0$ then $\bar{h}(X, Y)$ would be constant. Furthermore, it satisfies

$$(9) \quad (-b - 1 + XY - X^2) \frac{\partial \bar{h}}{\partial X} - \frac{\partial \bar{h}}{\partial Y} = 0.$$

Computing in (9) the coefficient of X^{n+1} we get

$$-n\bar{h}_n(Y) = 0 \quad \text{that is} \quad \bar{h}_n(Y) = 0,$$

a contradiction. This concludes the proof of the lemma. \square

To prove Proposition 3 we are left with the part of the Darboux polynomials with nonzero cofactor. To prove this part we will use the following result whose proof can be found in [3, 4].

Proposition 9. *We suppose that $h \in \mathbb{C}[x, y]$ and let $h = h_1^{n_1} \cdots h_r^{n_r}$ be its factorization in irreducible factors over $\mathbb{C}[x, y]$. Then for a polynomial system (2) $h = 0$ is an invariant algebraic curve with cofactor K_h if and only if $h_i = 0$ is an invariant algebraic curve for each $i = 1, \dots, r$ with cofactor $K_{f_{hi}}$. Moreover $K_h = n_1 K_{h_1} + \dots + n_r K_{h_r}$.*

Basically Proposition 9 states that to study the Darboux polynomials it is enough to consider the irreducible ones.

More precisely we will prove the following result.

Lemma 10. *System (2) has an irreducible Darboux polynomial $h = h(x, y)$ with non-zero cofactor if and only if $b = 0$, and in this case $h = y$ and the cofactor is $-x$.*

Proof. Let $h = h(x, y)$ be an irreducible Darboux polynomial of system (2) with non-zero cofactor. We introduce the change of variables

$$Y = y, \quad X = x + Y$$

and write system (2) in the form

$$(10) \quad X' = -1, \quad Y' = b - XY + Y^2,$$

that can be written as

$$\frac{dY}{dX} = -b + XY - Y^2.$$

Then $\bar{h} = \bar{h}(X, Y) = h(x, y)$ is an irreducible Darboux polynomial of system (10) with non-zero cofactor $\bar{k} = \alpha_0 + \alpha_1 X + \alpha_2 Y$ with $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{C}$ not all zero. Then \bar{h} satisfies

$$\frac{d\bar{h}}{dX} = \frac{\partial \bar{h}}{\partial X} + \frac{\partial \bar{h}}{\partial Y}(-b + YX - Y^2) = (\alpha_0 + \alpha_1 X + \alpha_2 Y)\bar{h},$$

that is

$$(11) \quad \frac{d\bar{h}}{dX} = (\alpha_0 + \alpha_1(X - Y) + (\alpha_1 + \alpha_2)Y)\bar{h}.$$

Now using that

$$X - Y = \frac{1}{Y} \frac{dY}{dX} + \frac{b}{Y},$$

we rewrite (11) as

$$\frac{d\bar{h}}{\bar{h}} = \alpha_0 dX + \frac{\alpha_1}{Y} dY + \left(\frac{b\alpha_1}{Y} + (\alpha_1 + \alpha_2)Y \right) dX,$$

that is, after integrating and taking exponentials,

$$\bar{h} = CY^{\alpha_1} \exp \left(\alpha_0 X + \left(\frac{b\alpha_1}{Y} + (\alpha_1 + \alpha_2)Y \right) X \right),$$

where C is a constant. Since \bar{h} must be a polynomial we have $\alpha_0 = 0$, $\alpha_1 + \alpha_2 = 0$ and $b\alpha_1 = 0$. Then the cofactor of \bar{h} is non-zero if and only if $b = 0$, and in this case \bar{h} is a polynomial of the form $\bar{h} = CY^{\alpha_1}$. Since \bar{h} is irreducible we get that $\alpha_1 = 1$. So we can take $\bar{h} = Y$ which yields $h = y$ and the cofactor $k = -x$. This completes the proof of the lemma. \square

Proof of Proposition 3. It follows directly from Lemmas 8 and 10. \square

4. PROOF OF PROPOSITION 4

To prove Proposition 4 we will use the following known result whose proof and geometrical meaning is given in [5].

Proposition 11. *The following statements hold.*

- (a) *If $E = \exp(g/h)$ is an exponential factor for the polynomial system (2) and h is not a constant polynomial, then $h = 0$ is an invariant algebraic curve.*
- (b) *Eventually e^g for $k = 1, 2, \dots$ can be exponential factors, coming from the multiplicity of the infinite invariant straight line.*

Let $E = \exp(g/h)$ be an exponential factor of system (2). In view of Propositions 3 and 11, we have that $E = \exp(g)$ if $b \neq 0$, and $E = \exp(g/y^n)$ with $n \geq 0$ if $b = 0$. When $n > 0$ then $y \nmid g$.

We first consider the case $E = \exp(g)$, $g \in \mathbb{C}[x, y]$. We have that the cofactor L has degree one. We introduce the change of variables of (6), then $g(x, y) = \bar{g}(X, Y)$ and \bar{g} satisfies

$$(12) \quad (-b - 1 + XY - X^2) \frac{\partial \bar{g}}{\partial X} - \frac{\partial \bar{g}}{\partial Y} = \beta_0 + \beta_1 X + \beta_2 Y,$$

for some $\beta_0, \beta_1, \beta_2 \in \mathbb{C}$. We write \bar{g} as a polynomial in the variable X as in (8). Computing in (12) the coefficient of X^{n+1} for $n \geq 1$ we get

$$-n\bar{g}_n(Y) = 0 \quad \text{that is} \quad \bar{g}_n(Y) = 0.$$

Therefore $\bar{g} = \bar{g}_0(Y)$. Imposing that it satisfies (12) we get

$$\frac{d\bar{g}_0}{dY} = -\beta_0 - \beta_1 Y - \beta_2 Y^2.$$

Solving it we get $\beta_1 = 0$ and $\bar{g} = -\beta_0 Y - \beta_2 Y^2/2$. So we have the exponential factor $\exp(Y)$ with cofactor -1 and $\exp(Y^2)$ with cofactor $-2Y$.

Now we assume $b = 0$ and that $E = \exp(\bar{g}/(Y - X)^n)$ with $n \geq 1$ and \bar{g} satisfies

$$(-1 + XY - X^2) \frac{\partial \bar{g}}{\partial X} - \frac{\partial \bar{g}}{\partial Y} = -nX\bar{g} + (\beta_0 + \beta_1 X + \beta_2 Y)(Y - X)^n,$$

for some $\beta_0, \beta_1, \beta_2 \in \mathbb{C}$. Denoting by $\hat{g} = \hat{g}(X) = \bar{g}(X, X)$ that is, evaluating it on $Y = X$ we get that it satisfies

$$\frac{d\hat{g}}{dX} = nX\hat{g}.$$

So $\hat{g} = Ce^{nX^2/2}$ with C a constant. Since \hat{g} is a polynomial it follows that $C = 0$. So $\hat{g} = 0$, and $Y - X$ divides $\bar{g}(X, Y)$. Consequently $y|g(x, y)$ a contradiction. This completes the proof of the proposition.

5. PROOF OF THEOREM 5

We first recall that a non-constant complex function $R: \mathbb{C}^2 \rightarrow \mathbb{C}$ is an *integrating factor* of the polynomial vector field χ on U , if one of the following three equivalent conditions holds

$$\frac{\partial(RP)}{\partial x} = -\frac{\partial(RQ)}{\partial y}, \quad \operatorname{div}(RP, RQ) = 0, \quad \chi R = -R \operatorname{div}(P, Q),$$

on U with $P = -b - 1 + xy$ and $Q = b - xy$. As usual the *divergence* of the vector field χ is given by

$$\operatorname{div}(P, Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = y - x.$$

From the Darboux theory of integrability we have the next result proved, for instance, in [4].

Theorem 12. *Suppose that the polynomial vector field χ of degree m defined in \mathbb{C}^2 admits p invariant algebraic curves $f_i = 0$ with cofactors K_i for $i = 1, \dots, p$, and q exponential factors $E_j = \exp(g_j/h_j)$ with $(g_j, h_j) = 1$ and cofactors L_j for $j = 1, \dots, q$. Then there exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that*

$$\sum_{i=1}^p \lambda_i K_i + \sum_{j=1}^q \mu_j L_j = -\operatorname{div}(P, Q),$$

if and only if the function of Darboux type

$$(13) \quad f_1^{\lambda_1} \dots f_p^{\lambda_p} E_1^{\mu_1} \dots E_q^{\mu_q}$$

is an integrating factor of the vector field χ .

To prove the results related with Liouvillian first integrals we use the following result proved in [2, 20].

Theorem 13. *The polynomial differential system (2) has a Liouvillian first integral if and only if it has an integrating factor of Darboux type (see (13)).*

By Proposition 3 system (2) has a Darboux polynomial if and only if $b = 0$ (in which case it is y with cofactor $K_1 = -x$), and by Proposition 4 it has the exponential factors $\exp(x+y)$ and $\exp((x+y)^2)$ with cofactors $L_1 = -1$ and $L_2 = -2(x+y)$, respectively.

In order that system (2) has a Liouvillian first integral, by Theorem 13, system (2) must have an integrating factor of Darboux type. From Theorem 12 system (2) has an integrating factor of Darboux type if and only if

$$\lambda_1 K_1 + \mu_1 L_1 + \mu_2 L_2 = -\lambda_1 x - \mu_1 - 2\mu_2(x+y) = -(y-x), \quad \lambda_1, \mu_1, \mu_2 \in \mathbb{C}.$$

This equality is possible if and only if $\mu_1 = 0$, $\lambda_1 = -2$ and $\mu_2 = -1/2$. Therefore, the integrating factor is $y^{-2}e^{(x+y)^2/2}$. Computing the first integral associated to this integrating factor we get the one stated in Theorem 5. This completes the proof of the theorem.

ACKNOWLEDGEMENTS

The first author is supported by the grants MICINN/FEDER MTM 2008-03437, AGAUR 2009SGR410 and by ICREA Academia. The third author is partially supported by FCT through CAMGDS, Lisbon.

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